Area

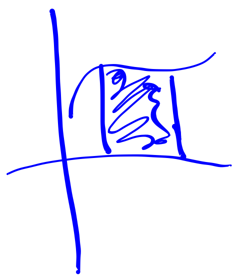
Approximate by sums of areas of overlaying rectangles

$\Downarrow$

Sigma Notation

$\Downarrow$

Area under a positive-valued function between $x=a$ and $x=b$ =



$$\lim_{n\to\infty} \sum_{k=1}^{n} \underbrace{f(x_k^*)\Delta x}_{\text{Area of rectangle}}$$

Antiderivative (one of many) = area function!

$\Downarrow$

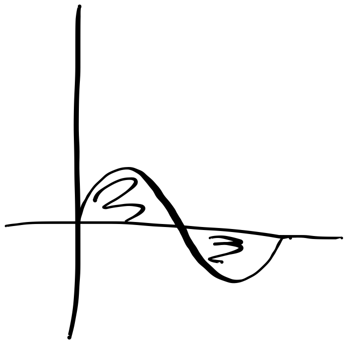
finding antideriv.

$\Downarrow$

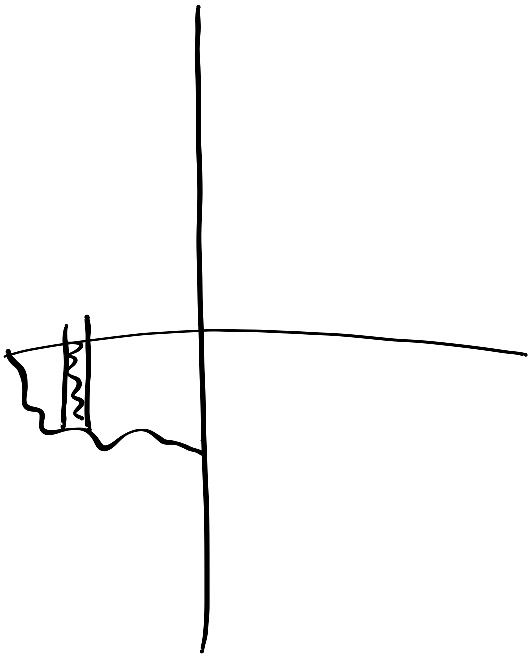
u-substitution

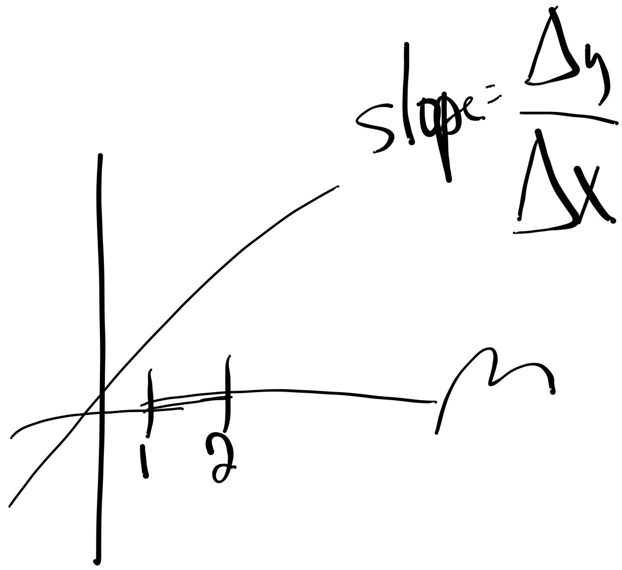
So the area "above" a negative-valued f^n =

$$\text{Area} = \lim_{n\to\infty} \sum_{k=1}^{n} |f(x_k^*)| \Delta x$$



Net signed area $= \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*) \Delta x$




slope = $\frac{\Delta y}{\Delta x}$
1
2

26) $$\lim_{n\to\infty} \sum_{k=1}^{n-1} \frac{2k^2}{n^3}$$

$$= \lim_{n\to\infty} \frac{2}{n^3}\left(\sum_{k=1}^{n-1} k^2\right)$$

$$\sum_{k=1}^{n-1} \frac{2k^2}{n^3} = \frac{2(1)}{n^3} + \frac{2(2^2)}{n^3} + \frac{2(3^2)}{n^3} + \dots + \frac{2(n-1)^2}{n^3}$$

$$\sum_{k=1}^{n} k^2 = \boxed{\frac{n(n+1)(2n+1)}{6}}$$

$$= \lim_{n\to\infty} \frac{2}{n^3}\left(\frac{(n-1)((n-1)+1)(2(n-1)+1)}{6}\right)$$

$$2(n-1)+1$$

$$= \lim_{n\to\infty} \frac{2}{n^3}\left(\frac{(n-1)(n)(2n-1)}{6}\right)$$

$$= \lim_{n\to\infty} \frac{1}{n^2}\left(\frac{(n-1)(2n-1)}{3}\right)$$

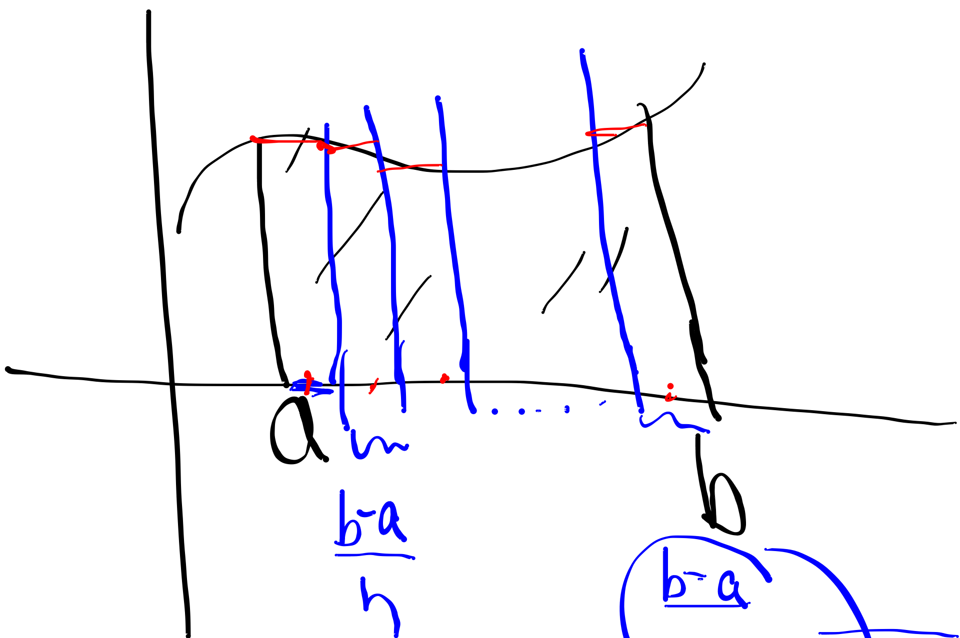
$$n-1 = n\left(1-\frac{1}{n}\right)$$

$$= \lim_{n\to\infty} \frac{1}{\cancel{n^2}}\left(\frac{\cancel{n^2}(1-\cancel{\tfrac{1}{n}}^{\,0})(2-\cancel{\tfrac{1}{n}}^{\,0})}{3}\right) = \frac{(1)(1)(2)}{(3)} = \frac{2}{3}$$

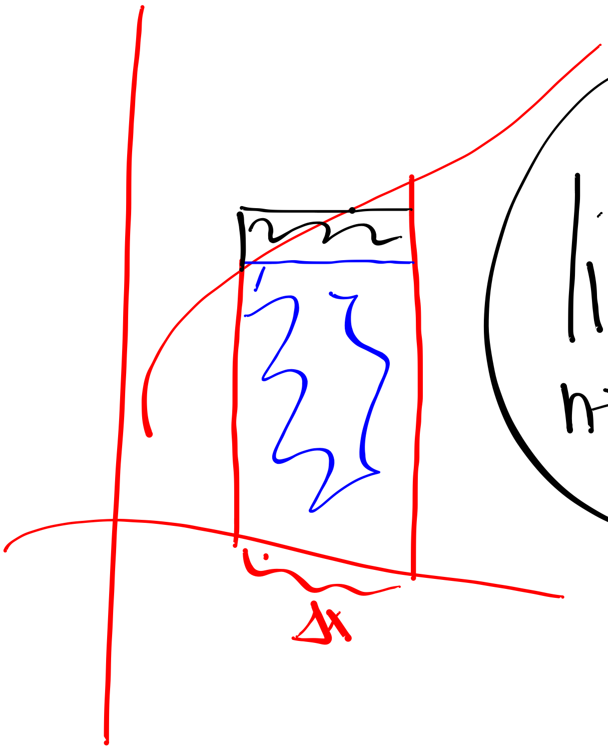
Area =

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x$$

a selected (possibly random?) x-value in that sub-interval



$$\frac{b-a}{n} = \Delta x$$



$$\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x$$

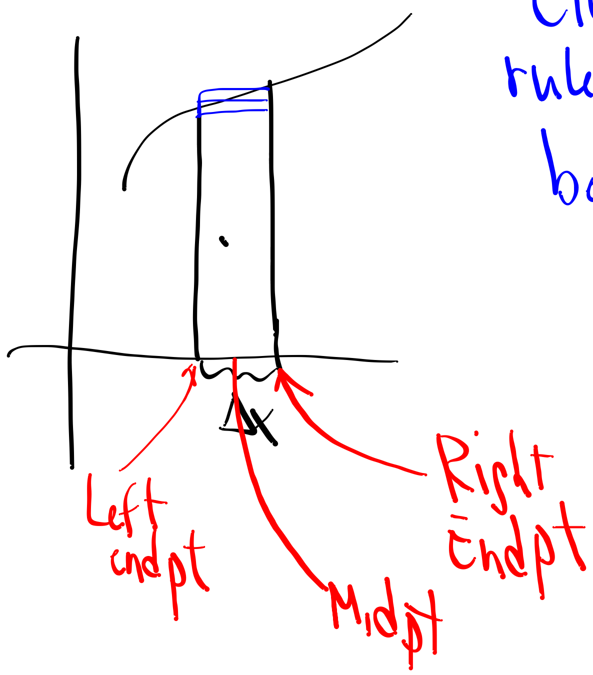
But when we APPROXIMATE an area there are some standard $f(x_k^*)$ we use . . .

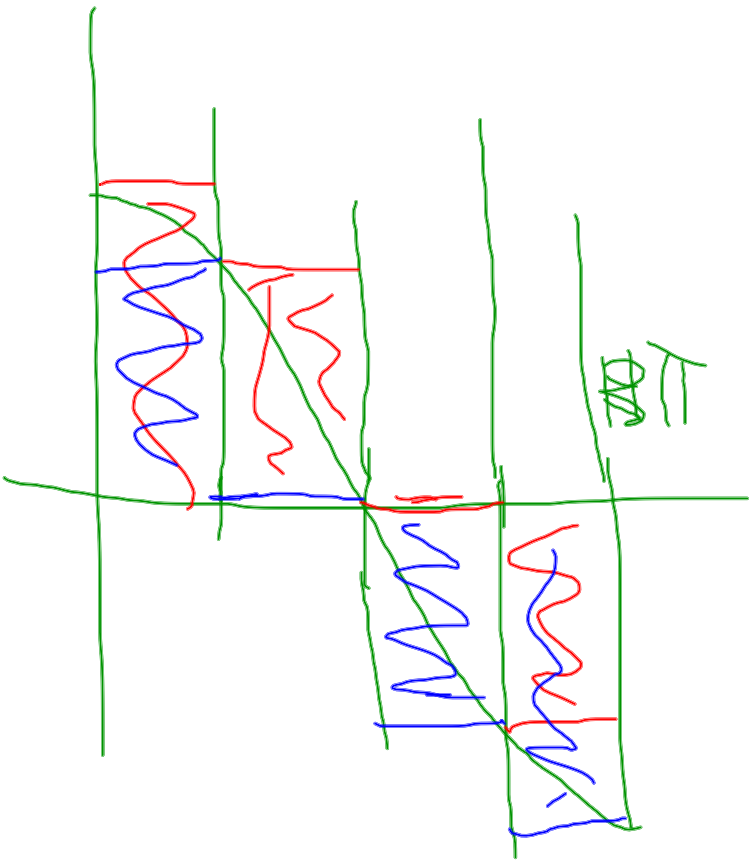
Left hand approximation : x_k^* is the x-value on the left boundary

Right hand approximation : x_k^* is . . . . right boundary

Midpoint approximation : x_k^* is the midpoint of the sub-interval

"closed form" rules are bottom of pg 396

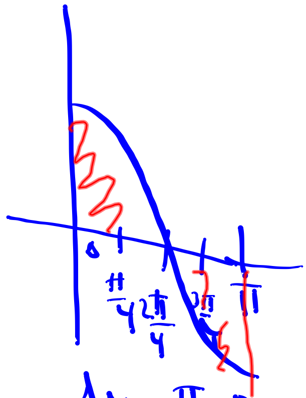




6.4/31

$f(x) = \cos x \; ; a = 0 \; ; b = \pi$

Compute $\displaystyle\sum_{k=1}^{4} f(x_k^*)\,\Delta x$



$\Delta x = \dfrac{\pi - 0}{4}$

Left endpt a) $\displaystyle\sum_{k=1}^{4} \sim =$

$$\cos\left(0\left(\tfrac{\pi}{4}\right)\right)\cdot\tfrac{\pi}{4} + \cos\left(1\left(\tfrac{\pi}{4}\right)\right)\cdot\tfrac{\pi}{4} + \cos\left(2\left(\tfrac{\pi}{4}\right)\right)\cdot\tfrac{\pi}{4} + \cos\left(3\left(\tfrac{\pi}{4}\right)\right)\cdot\tfrac{\pi}{4} =$$

mid pt b) $\Sigma =$

$$\cos\left(\tfrac{\pi}{8}\right)\cdot\tfrac{\pi}{4} + \cos\left(\tfrac{3\pi}{8}\right)\cdot\tfrac{\pi}{4} + \cos\left(\tfrac{5\pi}{8}\right)\cdot\tfrac{\pi}{4} + \cos\left(\tfrac{7\pi}{8}\right)\cdot\tfrac{\pi}{4} =$$

Right endpt c)

$$\cos\left(1\left(\tfrac{\pi}{4}\right)\right)\cdot\tfrac{\pi}{4} + \cos\left(2\left(\tfrac{\pi}{4}\right)\right)\cdot\tfrac{\pi}{4} + \cos\left(3\left(\tfrac{\pi}{4}\right)\right)\cdot\tfrac{\pi}{4} + \cos\left(4\left(\tfrac{\pi}{4}\right)\right)\cdot\tfrac{\pi}{4}$$

$$= \tfrac{\pi}{4}\left(\cos\left(1\left(\tfrac{\pi}{4}\right)\right) + \cos\left(2\left(\tfrac{\pi}{4}\right)\right) + \cdots + \cos\left(4\left(\tfrac{\pi}{4}\right)\right)\right)$$

Area

Approximate by sums of areas of overlaying rectangles

$\Downarrow$

Sigma notation

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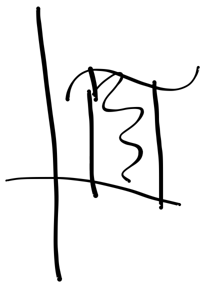
Area under positive-valued function between $x=a$ and $x=b$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \underbrace{f(x_k^*)}_{\text{height}} \underbrace{\Delta x}_{\text{width}}$$

Area of Rectangle



Antiderivative (one of many) is an area function!

$\Downarrow$

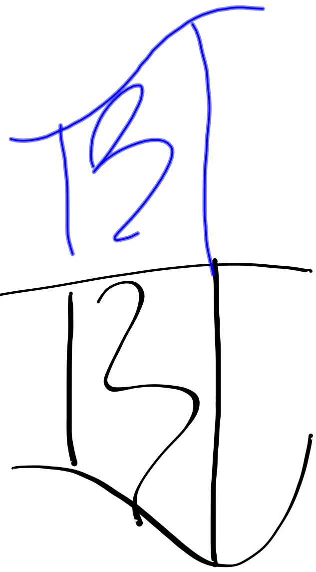
finding antiderivatives

$\Downarrow$

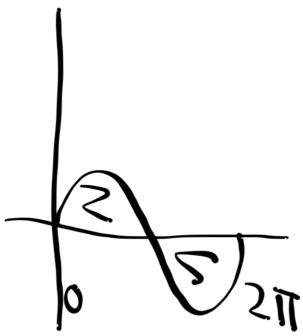
u-substitution

Area "above" neg-valued fn [between the x-axis & neg-valued f]

$$= \lim_{n\to\infty} \sum_{k=1}^{n} |f(x_k^*)| \Delta x$$



Net signed area $= \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*) \Delta x$



26)

$$\lim_{n\to\infty} \sum_{k=1}^{n-1} \frac{2k^2}{n^3}$$

$$\sum_{k=1}^{n-1} \frac{2k^2}{n^3} = \frac{2(1)^2}{n^3} + \frac{2(2)^2}{n^3} + \frac{2(3)^2}{n^3} + \cdots + \frac{2(n-1)^2}{n^3}$$

$$= \lim_{n\to\infty} \frac{2}{n^3} \sum_{k=1}^{n-1} k^2$$

$$= \lim_{n\to\infty} \frac{2}{n^3}\left[\frac{(n-1)(n-1+1)(2(n-1)+1)}{6}\right]$$

closed form formulas

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$= \lim_{n\to\infty} \frac{2}{n^3}\left(\frac{(n-1)(n)(2n-1)}{6}\right)$$

$$= \lim_{n\to\infty} \frac{2}{\cancel{n^3}}\left(\frac{\cancel{n \cdot n \cdot n}\,(1-\cancel{\tfrac{1}{n}}^{0})(2-\cancel{\tfrac{1}{n}}^{0})}{6}\right) = \frac{2(1)(2)}{6} = \frac{2}{3}$$

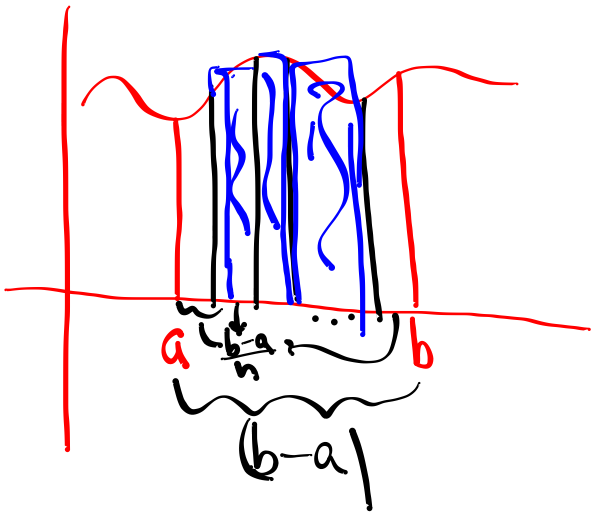
$$\sum_{k=1}^{n}(2k^2-k+4) = \sum_{k=1}^{n} 2k^2 - \sum_{k=1}^{n} k + \sum_{k=1}^{n} 4$$

$$= 2\sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} k + 4n$$

pick one x-value in the sub-interval

$$\text{Area} = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*)\,\Delta x$$

width of the sub-interval

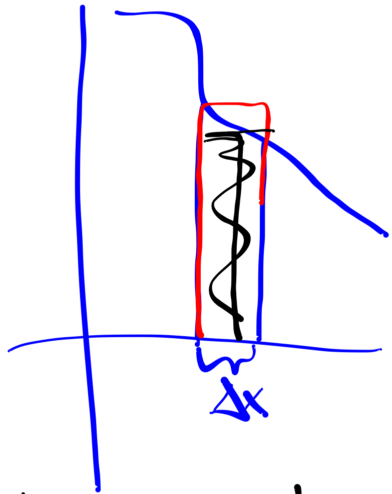


$$\frac{b-a}{n} = \Delta x$$

Approximations . .

come in 3

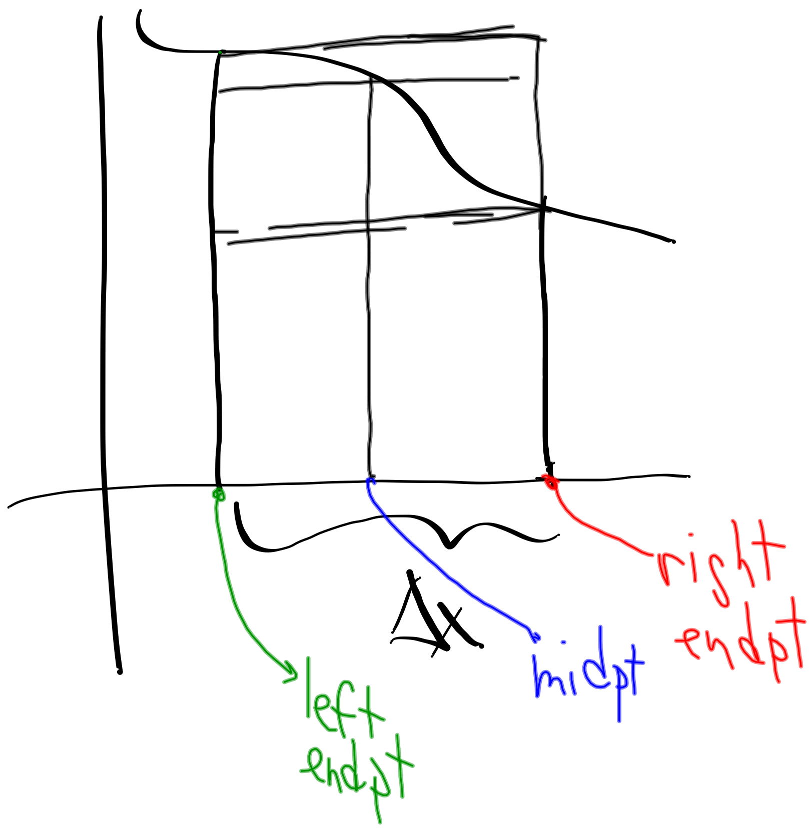
standard flavors



Left-endpt approx: x_k^* to be the left-most point.

Midpoint approx: x_k^* — midpt of sub-interval

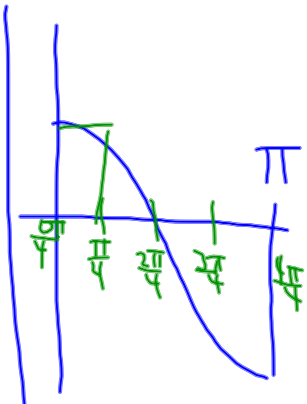
Right endpoint approx (right hand..) : x_k^* — right most point



Δx
left endpt
midpt
right endpt

6.4/31 $\quad f(x) = \cos x$; $a = 0$; $b = \pi$

find $\displaystyle\sum_{k=1}^{4} f(x_k^*)\Delta x$



$0 \longrightarrow \pi$

$\dfrac{\pi - 0}{4}$

a) use left hand approx

$$= \cos\left(0\left(\frac{\pi}{4}\right)\right)\cdot\frac{\pi}{4} + \cos\left(1\left(\frac{\pi}{4}\right)\right)\cdot\frac{\pi}{4} + \cos\left(2\left(\frac{\pi}{4}\right)\right)\cdot\frac{\pi}{4} + \cos\left(3\left(\frac{\pi}{4}\right)\right)\cdot\frac{\pi}{4}$$

$$= \frac{\pi}{4}\left(\cos(0) + \cos\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{2}\right) + \cos\left(\frac{3\pi}{4}\right)\right)$$

$$= \frac{\pi}{4}$$

b midpt

$$= \frac{\pi}{4}\left(\cos\left(\frac{\pi}{8}\right) + \cos\left(\frac{3\pi}{8}\right) + \cos\left(\frac{5\pi}{8}\right) + \cos\left(\frac{7\pi}{8}\right)\right)$$

$$= 0$$

c right hand approx

$$= \frac{\pi}{4}\left(\cos\left(\frac{\pi}{4}\right) + \cos\left(2\frac{\pi}{4}\right) + \cos\left(3\frac{\pi}{4}\right) + \cos\left(4\frac{\pi}{4}\right)\right)$$

$$= -\frac{\pi}{4}$$